

A TELESCOPE COMPARISON LEMMA FOR THH

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ABSTRACT. We extend to the non-connective case a lemma of Bökstedt about the equivalence of the telescope with a more complicated homotopy colimit of symmetric spectra used in the construction of THH.

1. INTRODUCTION

In [1], Bökstedt constructed the topological Hochschild homology of a symmetric ring spectrum, before the smash product of symmetric spectra had been invented. Now that we understand the smash product of symmetric spectra, we can construct THH using the cyclic bar construction. The paper [13] compares these definitions of THH. Both of these constructions are the geometric realizations of simplicial symmetric spectra that turn out to be stably equivalent in each simplicial level. A peculiarity of the theory of symmetric spectra is that not all stable equivalences are equivalences of the underlying prespectra, and simple examples show that these two constructions of THH do not always have weakly equivalent underlying prespectra.

The problem can be understood levelwise, where the ring structure plays no role. (The multiplication is only used to define the simplicial face maps.) Understanding simplicial level zero is the key to understanding all the levels, and we concentrate on this. As traditional in working with THH, instead of thinking in terms of prespectra, we can think in terms of functors from based spaces to based spaces. Then for a symmetric spectrum T , the zeroth level of the cyclic bar construction $thh_0(T)$ is just T itself, and this corresponds to the functor that sends a based space X to the space $\text{tel}_n \Omega^n(T_n \wedge X)$, that is, this space is the underlying infinite loop space of $T \wedge X$. The zeroth level of Bökstedt's construction is a homotopy colimit. It sends X to the space

$$THH_0(T; X) = \text{hocolim}_{n \in I} \Omega^n(T_n \wedge X).$$

Here I is the category finite sets and injections; see Definition 2.2. There is a canonical inclusion of the telescope in this homotopy colimit and Bökstedt's fundamental lemma for THH is that it is a weak equivalence when T is strictly connective ($\pi_q T_n = 0$ for $q < n$) and convergent (Definition 2.1). The purpose of this paper is to eliminate the connectivity hypothesis.

This work grew out of the authors' previous study of topological Hochschild homology as defined for symmetric spectra and for S -modules in [3] and [13]. This material also formed the basis of a first comparison of the homotopy categories of symmetric spectra, S -modules, and several other new symmetric monoidal categories of spectra carried out in a larger project with May and Schwede. See [11]

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for an expository account based on this earlier outline. This first approach has since been replaced by highly structured comparisons of model categories, see [7] and [12].

The definition of symmetric spectra easily generalizes to other, even non-topologically motivated contexts. For example, symmetric spectra have also been recently used to model stable \mathbb{A}^1 -local homotopy theory in the sense of [14], q.v. [5]. Although the proofs do not immediately transport to this setting, one expects similar statements to hold. In fact, the impetus for resurrecting this material came from questions posed by Voevodsky that are answered here.

We work in the setting of topological symmetric spectra, but only minor modifications, such as level fibrant replacement, would be necessary to apply these results to simplicial symmetric spectra. To avoid topological technicalities, we assume that any level of a given symmetric spectrum or any given based space here is non-degenerately based.

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2. STATEMENT OF RESULTS

We first need to recall a few definitions before we state our results.

Definition 2.1. Let T be a symmetric spectrum. It has an underlying prespectrum with spaces T_n and structure maps $\sigma_n : T_n \wedge S^1 \rightarrow T_{n+1}$. Define the homotopy groups of T to be the homotopy groups of the prespectrum:

$$\pi_q T = \operatorname{colim} \pi_{q+n} T_n \cong \pi_q(\operatorname{tel}_n \Omega^n T_n).$$

We say that T is *convergent* if the adjoint structure map $\tilde{\sigma}_n : T_n \rightarrow \Omega T_{n+1}$ is an $(n + \lambda(n))$ -equivalence for some nondecreasing sequence of integers $\lambda(n)$ with infinite limit. We say that T is a *symmetric Ω -spectrum* if the maps $\tilde{\sigma}_n$ are weak equivalences. We say that a map $f : T \rightarrow T'$ of symmetric spectra is a π_* -*isomorphism* if it induces an isomorphism of homotopy groups. We say that f is a *level weak equivalence* if each $f_n : T_n \rightarrow T'_n$ is a weak equivalence. Let $[X, Y]$ denote the set of maps $X \rightarrow Y$ in the homotopy category with respect to the level model structure [7, §6]. We say that f is a *stable weak equivalence* if $f^* : [Y, E] \rightarrow [X, E]$ is a bijection for all symmetric Ω -spectra E .

Recall from [4, 7] that a π_* -isomorphism is a stable weak equivalence, but not conversely. However, the converse does hold for stable weak equivalences between convergent spectra, and we shall only be concerned with convergent spectra here.

Definition 2.2. Let I be the category whose objects are the sets $\mathbf{n} = \{1, \dots, n\}$, $n \geq 0$, and whose morphisms are the injective functions. Let Σ be the subcategory of isomorphisms in I . For $m < n$, let $i_{m,n} : \mathbf{m} \rightarrow \mathbf{n}$ be the standard inclusion and observe that any map $f : \mathbf{m} \rightarrow \mathbf{n}$ can be written, non-uniquely, as a composite $f = \tau \circ i_{m,n}$ for some $\tau \in \Sigma_n$. Let $J \subset I$ be the subcategory of objects \mathbf{n} and standard injections $i_{m,n} : \mathbf{m} \rightarrow \mathbf{n}$ for $m \leq n$.

Our main theorem is the following sharpening of a basic lemma of Bökstedt [1, 1.6] that eliminates its connectivity hypotheses; see also [13, 3.1.7] for a further generalization in the case $X = S^0$. Let \mathcal{T} be the category of based spaces. A sequence of based spaces X_n and maps $X_n \rightarrow X_{n+1}$ determines a functor $J \rightarrow \mathcal{T}$, and there is a natural homotopy equivalence

$$\mathrm{tel}_n X_n \xrightarrow{\simeq} \mathrm{hocolim}_J X_n.$$

A functor defined on I restricts to a functor defined on J and thus gives an induced map of homotopy colimits.

A symmetric spectrum T gives rise to a functor $I \rightarrow \mathcal{T}$ that sends \mathbf{n} to $\Omega^n(T_n \wedge X)$. Using the evident natural map $\Omega^n(X) \wedge Y \rightarrow \Omega^n(X \wedge Y)$, the adjoint structure map $\tilde{\sigma}$ gives

$$T_m \wedge X \rightarrow (\Omega^{n-m} T_n) \wedge X \rightarrow \Omega^{n-m}(T_n \wedge X);$$

the map induced by $i_{m,n}$ is obtained from this by applying Ω^m . Permutations act on S^n and T_n and by conjugation on $\Omega^n(T_n \wedge X)$. This construction plays a fundamental role in the theory of symmetric spectra, see [13].

Theorem 2.3 (Bökstedt's lemma). *Let T be a convergent symmetric spectrum and X be a based space. Then the natural map*

$$\mathrm{tel}_n \Omega^n(T_n \wedge X) \simeq \mathrm{hocolim}_J \Omega^n(T_n \wedge X) \rightarrow \mathrm{hocolim}_I \Omega^n(T_n \wedge X)$$

is a weak equivalence.

A proof of Bökstedt's original lemma appears in [6, 2.3.7]. When the symmetric spectrum T is not assumed to be connective, the homotopy groups of $T_n \wedge X$ do not stabilize, and the proof of Theorem 2.3 requires the construction and analysis of $(-k)$ -connected covers; see Lemma 4.3. Bökstedt and Madsen's argument applies on each cover, and we use standard techniques with homotopy colimits to mesh them together and deduce the result for T . For the reader's convenience we review the definition and basic properties of homotopy colimits in Section 6.

It is a standard fact in stable homotopy theory that the map

$$\mathrm{tel}_n \Omega^n(T_n \wedge X) \rightarrow \Omega(\mathrm{tel}_n \Omega^n(T_n \wedge \Sigma X))$$

is a weak equivalence; see for example [7, 7.4.i']. It follows that when T is convergent, the map

$$\mathrm{hocolim}_I \Omega^n(T_n \wedge X) \rightarrow \Omega(\mathrm{hocolim}_I \Omega^n(T_n \wedge \Sigma X))$$

is a weak equivalence. We obtain the following corollary of Theorem 2.3.

Corollary 2.4. *If T is convergent, then, for a based space X , we have an isomorphism of homology*

$$T_q(X) = \pi_q(T \wedge X) \cong \pi_{q+k} \mathrm{hocolim}_I \Omega^n(T_n \wedge \Sigma^k X)$$

and an isomorphism of cohomology

$$T^q(X) = \pi_{-q}(F(X, T)) \cong \pi_{k-q} F(\Sigma^{k-q} X, \mathrm{hocolim}_I \Omega^n(T_n \wedge S^k)).$$

For S the sphere spectrum, $S \wedge X$ is isomorphic to $\Sigma^\infty X$, and we obtain the following corollary. Recall that $QX = \mathrm{colim} \Omega^n \Sigma^n X$. Since the maps of the colimit system are inclusions, the natural map $\mathrm{tel}_n \Omega^n \Sigma^n X \rightarrow QX$ is a weak equivalence.

Corollary 2.5. *The natural map*

$$\mathrm{tel}_n \Omega^n \Sigma^n X \simeq \mathrm{hocolim}_I \Omega^n \Sigma^n X \longrightarrow \mathrm{hocolim}_I \Omega^n \Sigma^n X$$

is a weak equivalence.

3. THE PROOF OF BÖKSTEDT'S LEMMA

The strategy for the proof of Theorem 2.3 is to reduce to the study of a symmetric Ω -spectrum that is closely related to $T \wedge X$. Let $\mathbb{Q}_X T$ be the symmetric spectrum with levels

$$\mathbb{Q}_X T_n = \mathrm{tel}_m \Omega^m (T_{n+m} \wedge X).$$

Then \mathbb{Q}_X is a functor of both T and X . The inclusions of the initial term in a telescope gives a natural map of symmetric spectra $T \wedge X \longrightarrow \mathbb{Q}_X T$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{tel} \Omega^n (T_n \wedge X) & \longrightarrow & \mathrm{tel} \Omega^n \mathbb{Q}_X T_n \\ \downarrow & & \downarrow \\ \mathrm{hocolim}_I \Omega^n (T_n \wedge X) & \longrightarrow & \mathrm{hocolim}_I \Omega^n \mathbb{Q}_X T_n. \end{array}$$

Theorem 2.3 states that the left vertical arrow is a weak equivalence. To prove this, we prove that the remaining three arrows are weak equivalences. We prove the following two lemmas in the next section.

Lemma 3.1. *If T is a convergent symmetric spectrum, then $\mathbb{Q}_X T$ is a symmetric Ω -spectrum and the natural map $T \wedge X \longrightarrow \mathbb{Q}_X T$ is a π_* -isomorphism.*

For $q \geq 0$, the map of q th homotopy groups of spaces induced by the top arrow in the diagram is the map of q th homotopy groups of prespectra induced by $T \wedge X \longrightarrow \mathbb{Q}_X T$, so this lemma implies directly that the top arrow is a weak equivalence.

Warning 3.2. If T is not convergent, then $\mathbb{Q}_X T$ is not a symmetric Ω -spectrum and $T \wedge X \longrightarrow \mathbb{Q}_X T$ is not a π_* -isomorphism in general, even when $X = S^0$. In fact, it is shown in [4] that the symmetric spectrum $T = F_1(S^1)$ defined below gives a counterexample.

The next lemma shows that the bottom horizontal arrow in the diagram is a weak equivalence.

Lemma 3.3. *If T is a convergent symmetric spectrum, then the natural map*

$$\mathrm{hocolim}_I \Omega^n (T_n \wedge X) \longrightarrow \mathrm{hocolim}_I \Omega^n \mathbb{Q}_X T_n$$

is a weak equivalence.

Finally, the following lemma allows us to exploit the fact that $\mathbb{Q}_X T$ is a symmetric Ω -spectrum to prove that the right vertical arrow in the diagram is a weak equivalence.

Lemma 3.4. *If U is a symmetric Ω -spectrum, then the natural map*

$$U_0 \longrightarrow \mathrm{hocolim}_I \Omega^n U_n$$

is a weak equivalence.

This lemma follows from a standard result about homotopy colimits of homotopically constant diagrams, see Lemma 6.2.

By Lemmas 3.1 and 3.4, when T is convergent the inclusion

$$\mathbb{Q}_X T_0 \longrightarrow \operatorname{hocolim}_I \Omega^n \mathbb{Q}_X T_n$$

is a weak equivalence. The inclusion

$$\mathbb{Q}_X T_0 \longrightarrow \operatorname{tel} \Omega^n \mathbb{Q}_X T_n$$

is also a weak equivalence by Lemma 6.2. Since the right vertical arrow is a map under $\mathbb{Q}_X T_0$, it too is a weak equivalence. This completes the proof of Theorem 2.3, modulo the proofs of the lemmas.

4. THE PROOFS OF TWO TECHNICAL LEMMAS

The proofs of Lemmas 3.1 and 3.3 are based on the use of “strictly k -connected covers” of symmetric spectra.

Definition 4.1. A symmetric spectrum T is *strictly k -connected* if T_n is $(n+k)$ -connected for $n \geq 0$.

We prove the following lemma in the next section.

Lemma 4.2. *For each integer k , there is a functor C_k on the category of symmetric spectra such that, for any symmetric spectrum T , $C_k T$ is strictly k -connected. There are natural transformations $c_k : C_k \longrightarrow \operatorname{Id}$ and $e_k : C_k \longrightarrow C_{k-1}$ such that $c_{k-1} = c_k \circ e_k$ and c_k induces an isomorphism $\pi_i C_k T_n \longrightarrow \pi_i T_n$ for $i > n+k$.*

The following observation is the key to the proofs of Lemmas 3.1 and 3.3.

Lemma 4.3. *Let T be a convergent symmetric spectrum. For each integer k and based space X , $C_k T \wedge X$ is a convergent symmetric spectrum.*

Proof. Assume that the structure map $\tilde{\sigma} : T_n \longrightarrow \Omega T_{n+1}$ is an $(n + \lambda(n))$ -equivalence, where $\lambda(n)$ is a nondecreasing sequence with infinite limit. The structure maps of $C_k T$ then have the same property. An easy diagram chase shows that the n th structure map of $C_k T \wedge X$ factors as the composite

$$C_k T_n \wedge X \xrightarrow{\tilde{\sigma} \wedge \operatorname{Id}} (\Omega C_k T_{n+1}) \wedge X \longrightarrow \Omega(C_k T_{n+1} \wedge X).$$

The first map is an $(n + \lambda(n))$ -equivalence and for $n+k > 1$ the second map is a $(2n + 2k + 1)$ -equivalence, by Lemma 4.4 below. The composite is therefore an $(n + \min(n + 2k + 1, \lambda(n)))$ -equivalence. \square

Lemma 4.4. *If Y is r -connected, then the canonical map $(\Omega Y) \wedge X \longrightarrow \Omega(Y \wedge X)$ is a $(2r - 1)$ -equivalence.*

Proof. This follows from the Freudenthal suspension theorem applied to $\Omega Y \wedge X$ and to ΩY , together with the factorization of the cited map as the composite

$$\Omega Y \wedge X \xrightarrow{\eta} \Omega \Sigma(\Omega Y \wedge X) \cong \Omega(\Sigma \Omega Y \wedge X) \xrightarrow{\Omega(\varepsilon \wedge \operatorname{Id})} \Omega(Y \wedge X),$$

where η and ε are the unit and counit of the (Σ, Ω) -adjunction. \square

We need the following general result about homotopy colimits over I to prove Lemmas 3.1 and 3.3.

Proposition 4.5. *Let F and F' be functors $I \rightarrow \mathcal{T}$ and let $\phi : F \rightarrow F'$ be a natural transformation. Assume that $\phi(\mathbf{n}) : F(\mathbf{n}) \rightarrow F'(\mathbf{n})$ is a $\lambda(n)$ -equivalence, where $\lambda(n) \leq \lambda(n+1)$ and $\lim_n \lambda(n) = \infty$. Then the induced map*

$$\phi_* : \operatorname{hocolim}_I F \rightarrow \operatorname{hocolim}_I F'$$

is a weak equivalence.

The proof of this proposition is delayed to section 6 where we discuss homotopy colimits.

The following simple observation is used several times in the following proofs of Lemmas 3.1 and 3.3.

Lemma 4.6. *For any sequence of based spaces Y_n and based maps $Y_n \rightarrow Y_{n+1}$, the evident maps $\Omega Y_n \rightarrow \Omega \operatorname{tel}_n Y_n$ induce a weak equivalence*

$$\operatorname{tel}_n \Omega Y_n \rightarrow \Omega \operatorname{tel}_n Y_n.$$

Proofs of Lemmas 3.1 and 3.3. We are given a convergent symmetric spectrum T . We first show that both lemmas hold when T is replaced by $C_k T$ for any fixed integer k . We show that $\mathbb{Q}_X C_k T$ is a symmetric Ω -spectrum, and the map

$$(4.7) \quad C_k T \wedge X \rightarrow \mathbb{Q}_X C_k T$$

is a π_* -isomorphism.

The group $\pi_j \mathbb{Q}_X C_k T_n$ is the colimit of the sequence

$$\pi_j(C_k T_n \wedge X) \longrightarrow \pi_{j+1}(C_k T_{n+1} \wedge X) \longrightarrow \pi_{j+2}(C_k T_{n+2} \wedge X) \longrightarrow \cdots$$

We can calculate the maps $\pi_j \mathbb{Q}_X C_k T_n \rightarrow \pi_{j+1} \mathbb{Q}_X C_k T_{n+1}$ induced by the structure maps $\mathbb{Q}_X C_k T_n \rightarrow \Omega(\mathbb{Q}_X C_k T_{n+1})$ as maps of colimits arising from the diagrams

$$\begin{array}{ccccc} \pi_j(C_k T_n \wedge X) & \longrightarrow & \pi_{j+1}(C_k T_{n+1} \wedge X) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \pi_{j+1}(C_k T_{n+1} \wedge X) & \longrightarrow & \pi_{j+2}(C_k T_{n+2} \wedge X) & \longrightarrow & \cdots \end{array}$$

Since $C_k T \wedge X$ is convergent, the vertical maps in each such diagram are eventually isomorphisms and so induce isomorphisms of the colimits. Thus each structure map $\mathbb{Q}_X C_k T_n \rightarrow \Omega \mathbb{Q}_X C_k T_{n+1}$ is a weak equivalence. The map on homotopy groups induced by (4.7) is the map

$$\operatorname{colim}_n \pi_{j+n}(C_k T_n \wedge X) \rightarrow \operatorname{colim}_{m,n} \pi_{j+n+m}(C_k T_{n+m} \wedge X)$$

obtained by mapping the terms in the source to the terms with $m = 0$ in the target. Extending the diagram above vertically and arguing similarly, we see that this map is an isomorphism. We should note here that the subtlety mentioned in Warning 3.2 appears because the vertical maps here differ from the horizontal maps up to isomorphism, see [4, 5.6.3]. But this difficulty is avoided here since $C_k T \wedge X$ is convergent.

For Lemma 3.3, since $C_k T \wedge X$ is convergent the colimit, $\operatorname{colim} \pi_{i+k}(C_k T_{n+k} \wedge X)$, is attained at $\pi_i(C_k T_n \wedge X)$ for $i \leq n + \lambda(n)$. This directly implies the hypotheses required by Proposition 4.5 to show that

$$(4.8) \quad \operatorname{hocolim}_I \Omega^n(C_k T_n \wedge X) \rightarrow \operatorname{hocolim}_I \Omega^n \mathbb{Q}_X C_k T_n$$

is a weak equivalence.

We now deduce that the lemmas hold for T . We first show that $\mathbb{Q}_X T$ is a symmetric Ω -spectrum. Using the maps $C_{-k}T \rightarrow C_{-(k+1)}T$, we define $T' = \text{tel}_k C_{-k}T$. The compatible maps $C_{-k}T \rightarrow T$ induce a level weak equivalence of symmetric spectra $c : T' \rightarrow T$. Observe that $T' \wedge X \cong \text{tel}_k(C_{-k}T \wedge X)$. Similarly, using the induced maps $\mathbb{Q}_X C_{-k}T \rightarrow \mathbb{Q}_X C_{-(k+1)}T$, we define $\mathbb{Q}'_X T = \text{tel}_k \mathbb{Q}_X C_{-k}T$. The compatible maps $\mathbb{Q}_X C_{-k}T \rightarrow \mathbb{Q}_X T$ induce a map of symmetric spectra $d : \mathbb{Q}'_X T \rightarrow \mathbb{Q}_X T$ such that the following diagram commutes:

$$\begin{array}{ccc} T' \wedge X & \longrightarrow & \mathbb{Q}'_X T \\ c \wedge \text{id} \downarrow & & \downarrow d \\ T \wedge X & \longrightarrow & \mathbb{Q}_X T. \end{array}$$

The top horizontal arrow is a π_* -isomorphism since it is obtained by passage to telescopes from the π_* -isomorphisms (4.7). The map $c \wedge \text{id}$ is a level weak equivalence since c is a level weak equivalence. Using Lemma 4.6, we see that $\mathbb{Q}'_X T$ is a symmetric Ω -spectrum since each $\mathbb{Q}_X C_k T$ is a symmetric Ω -spectrum. Thus it suffices to show that d is a level weak equivalence to conclude both that $\mathbb{Q}_X T$ is a symmetric Ω -spectrum and that the bottom horizontal arrow in the diagram is a π_* -isomorphism, giving Lemma 3.1. On passage to telescopes, the weak equivalences

$$\text{tel}_k \Omega^m(C_{-k}T_{n+m} \wedge X) \rightarrow \Omega^m \text{tel}_k C_{-k}T_{n+m} \wedge X$$

given by Lemma 4.6 induce a weak equivalence

$$\begin{aligned} \mathbb{Q}'_X T &\cong \text{tel}_m \text{tel}_k \Omega^m(C_{-k}T_{n+m} \wedge X) \\ &\rightarrow \text{tel}_m \Omega^m(\text{tel}_k C_{-k}T_{n+m} \wedge X) = \mathbb{Q}_X T'. \end{aligned}$$

This is the n th term of a level weak equivalence $b : \mathbb{Q}'_X T \rightarrow \mathbb{Q}_X T'$. It is easily seen that d factors as the composite

$$\mathbb{Q}'_X T \xrightarrow{b} \mathbb{Q}_X T' \xrightarrow{\mathbb{Q}_X c} \mathbb{Q}_X T.$$

Since c is a level weak equivalence, so is $\mathbb{Q}_X c$. Therefore d is a level weak equivalence. This completes the proof of Lemma 3.1.

Finally, to complete the proof of Lemma 3.3, we observe that the level weak equivalences c and d induce the maps \bar{c} and \bar{d} displayed in the following commutative diagram:

$$\begin{array}{ccc} \text{tel}_k \text{hocolim}_I \Omega^n(C_k T_n \wedge X) & \longrightarrow & \text{tel}_k \text{hocolim}_I \Omega^n \mathbb{Q}_X C_k T_n \\ \cong \downarrow & & \downarrow \cong \\ \text{hocolim}_I \text{tel}_k \Omega^n(C_k T_n \wedge X) & \longrightarrow & \text{hocolim}_I \text{tel}_k \Omega^n \mathbb{Q}_X C_k T_n \\ \simeq \downarrow & & \downarrow \simeq \\ \text{hocolim}_I \Omega^n T'_n \wedge X & \longrightarrow & \text{hocolim}_I \Omega^n(\mathbb{Q}'_X T_n) \\ \bar{c} \downarrow & & \downarrow \bar{d} \\ \text{hocolim}_I \Omega^n(T_n \wedge X) & \longrightarrow & \text{hocolim}_I \Omega^n \mathbb{Q}_X T_n. \end{array}$$

The top horizontal arrow is a weak equivalence because it is the telescope of the weak equivalences (4.8). The vertical arrows labelled \cong are homeomorphisms obtained by commuting tel_k with hocolim_I . The vertical arrows labelled \simeq are weak

equivalences induced by weak equivalences of Lemma 4.6. The maps \bar{c} and \bar{d} are weak equivalences since c and d are level weak equivalences. Therefore the bottom horizontal arrow is a weak equivalence, which is the conclusion of Lemma 3.3. \square

5. POSTNIKOV TOWERS AND CONNECTIVE COVERS OF SYMMETRIC SPECTRA

We construct the strict coverings promised in Lemma 4.2 from the functorial strict Postnikov towers given by the following lemma.

Lemma 5.1. *For each integer k , there is a functor P_k on the category of symmetric spectra and a natural transformation $\xi_k : \text{Id} \longrightarrow P_k$ such that for every symmetric spectrum T and every n , the map*

$$\xi_k(\mathbf{n}) : T_n \longrightarrow P_k T_n$$

is a $(k+n+1)$ -equivalence and $\pi_i P_k T_n = 0$ for $i > k+n$. Further, there are natural transformations $\chi_k : P_k \longrightarrow P_{k-1}$ such that $\xi_{k-1} = \chi_k \circ \xi_k$.

Proof of Lemma 4.2. We define the symmetric spectrum $C_k T$ to be the (levelwise) homotopy fiber of the map ξ_k . For each \mathbf{n} in Σ , we define $c_k : C_k T_n \longrightarrow T_n$ to be the left vertical arrow in the following pullback diagram:

$$\begin{array}{ccc} C_k T_n & \longrightarrow & \text{map}_{\mathcal{T}}(I, P_k T_n) \\ c_k \downarrow & & \downarrow \\ T_n & \xrightarrow{\xi_k(\mathbf{n})} & P_k T_n. \end{array}$$

It is clear from the functoriality of our Postnikov sections that C_k defines a functor on symmetric spectra and that c_k is natural. Since $\xi_k(\mathbf{n})$ is a $(k+n+1)$ -equivalence, $C_k T$ is strictly k -connected. Since $\pi_i P_k T_n = 0$ for $i > k+n$, $c_k(\mathbf{n})$ induces an isomorphism on homotopy groups in degrees $k+n+1$ and above. The maps e_k are induced by the maps χ_k , and the equation $c_{k-1} = c_k \circ e_k$ follows from the corresponding equation $\xi_{k-1} = \chi_k \circ \xi_k$. \square

We will prove Lemma 5.1 by a localization technique analogous to one that allows a direct construction of functorial Postnikov towers in the category of spaces. We will need the left adjoint F_n to the n th space functor from symmetric spectra to spaces. Thus we have homeomorphisms of hom spaces

$$\text{map}_{Sp^{\Sigma}}(F_n X, T) \cong \text{map}_{\mathcal{T}}(X, T_n).$$

As noted in [7, 4.2], F_n is given explicitly by

$$(F_n X)_m = \Sigma_{m+} \wedge_{\Sigma_{m-n}} X \wedge S^{m-n},$$

where we interpret S^{m-n} as $*$ for $m < n$.

We construct P_k using the “small objects argument” from model category theory. For each n , define a functor $L_{k,n}$ on the category of symmetric spectra and a natural transformation $\eta : \text{Id} \longrightarrow L_{k,n}$ as follows. For a symmetric spectrum T , let $D_{k,n} T$ be the set of maps of symmetric spectra $F_n S^{k+n+1} \longrightarrow T$. Define $L_{k,n} T$ and η by

the following pushout diagram in the category of symmetric spectra:

$$\begin{array}{ccc} \bigvee_{f \in D_{k,n}T} F_n S^{k+n+1} & \xrightarrow{\vee f} & T \\ \downarrow \vee F_n i & & \downarrow \eta \\ \bigvee_{f \in D_{k,n}T} F_n C S^{k+n+1} & \longrightarrow & L_{k,n} T. \end{array}$$

Here i denotes the inclusion of the sphere in the cone.

When $m < n$, the space $F_n S_m^{n+k+1}$ is a point and so $\eta(\mathbf{m}) : T_m \rightarrow L_{k,n} T_m$ is an isomorphism. When $m \geq n$, the space $F_n S_m^{n+k+1}$ is a wedge of S^{m+k+1} 's, and so the map $\eta(\mathbf{m}) : T_m \rightarrow L_{k,n} T_m$ induces an isomorphism on homotopy groups in degrees 0 to $m+k$ inclusive. Moreover, by adjunction, $\eta(\mathbf{n})$ induces the zero map $\pi_{n+k+1} T_n \rightarrow \pi_{n+k+1} L_{k,n} T_n$. Let $N_{k,n}$ be the telescope of iterates of η :

$$N_{k,n} = \text{tel}(\text{Id} \xrightarrow{\eta} L_{k,n} \xrightarrow{\eta} L_{k,n} \circ L_{k,n} \xrightarrow{\eta} L_{k,n} \circ L_{k,n} \circ L_{k,n} \xrightarrow{\eta} \cdots).$$

Then $\pi_{n+k+1} N_{k,n} T_n = 0$ and there is a natural transformation $\nu_{k,n} : \text{Id} \rightarrow N_{k,n}$ such that $\nu_{k,n}(\mathbf{m}) : T_m \rightarrow N_{k,n} T_m$ induces an isomorphism on homotopy groups in degrees 0 to $m+k$ inclusive.

Let N_k be the telescope of the maps $\nu_{k,n}$:

$$N_k = \text{tel}(\text{Id} \xrightarrow{\nu_{k,0}} N_{k,0} \xrightarrow{\nu_{k,1}} N_{k,1} \circ N_{k,0} \xrightarrow{\nu_{k,2}} N_{k,2} \circ N_{k,1} \circ N_{k,0} \xrightarrow{\nu_{k,3}} \cdots).$$

Then $\pi_{n+k+1} N_k T_n = 0$ for all n and there is a natural transformation $\nu_k : \text{Id} \rightarrow N_k$ such that $\nu_k(\mathbf{n}) : T_n \rightarrow N_k T_n$ induces an isomorphism on homotopy groups in degrees 0 to $n+k$ inclusive for all n .

Since the map ν_k is the inclusion of the initial object in a telescope, each map $\nu_k(\mathbf{n}) : T_n \rightarrow N_k T_n$ is a cofibration. Let P_k be the colimit of the levelwise cofibrations ν_j for $j > k$:

$$P_k = \text{colim}(N_k \xrightarrow{\nu_{k+1}} N_{k+1} \circ N_k \xrightarrow{\nu_{k+2}} N_{k+2} \circ N_{k+1} \circ N_k \xrightarrow{\nu_{k+3}} \cdots).$$

There is a natural transformation $\xi_k : \text{Id} \rightarrow P_k$ induced by ν_k . It follows from the properties of N_k that each space $P_k T_n$ has zero homotopy groups above degree $n+k$ and that each map $\xi_k(\mathbf{n})$ is a $(k+n+1)$ -equivalence. There are natural transformations $\chi_k : P_k \rightarrow P_{k-1}$ induced by the following map of diagrams:

$$\begin{array}{ccccc} N_k & \xrightarrow{\nu_{k+1}} & N_{k+1} \circ N_k & \xrightarrow{\nu_{k+2}} & \cdots \\ \downarrow N_k \nu_{k-1} & & \downarrow N_{k+1} N_k \nu_{k-1} & & \\ N_{k-1} & \xrightarrow{\nu_k} & N_k \circ N_{k-1} & \xrightarrow{\nu_{k+1}} & N_{k+1} \circ N_k \circ N_{k-1} \xrightarrow{\nu_{k+2}} \cdots \end{array}$$

The diagram commutes because of the naturality of the horizontal maps. We see that $\chi_k \circ \xi_k = \xi_{k-1}$ since the following diagram commutes:

$$\begin{array}{ccc} \text{Id} & \xrightarrow{\nu_k} & N_k \\ \downarrow \nu_{k-1} & & \downarrow N_k \nu_{k-1} \\ N_{k-1} & \xrightarrow{\nu_k} & N_k \circ N_{k-1}. \end{array}$$

The functors P_k and natural transformations ξ_k and χ_k have the properties specified in Lemma 5.1.

6. HOMOTOPY COLIMITS OF SPACES AND SYMMETRIC SPECTRA

We first describe what we have used about homotopy colimits of spaces.

Let \mathcal{C} be a small discrete category, thought of as an indexing category. For a functor $F : \mathcal{C} \rightarrow \mathcal{T}$, the homotopy colimit of F is defined in terms of the one-sided categorical bar construction $B(*, -, -)$ as

$$\mathrm{hocolim}_{\mathcal{C}} F = B(*, \mathcal{C}, F).$$

See [3, X.3] or [10, §12]. The same definition is given in different language in [2, XII§2]. This is the geometric realization of a simplicial space, and the inclusion of its subspace of zero simplices gives a map

$$\bigvee_{c \in \mathrm{Ob} \mathcal{C}} F(c) \rightarrow \mathrm{hocolim}_{\mathcal{C}} F.$$

Homotopy colimits are functorial in F : a natural transformation $\phi : F \rightarrow F'$ of functors $\mathcal{C} \rightarrow \mathcal{T}$ induces a map

$$\phi_* : \mathrm{hocolim}_{\mathcal{C}} F \rightarrow \mathrm{hocolim}_{\mathcal{C}} F'.$$

The following main property of homotopy colimits, that they are homotopy invariant, can be proven as in [9, A.4].

Lemma 6.1. *If ϕ is an levelwise n -equivalence, weak equivalence, or homotopy equivalence, then ϕ_* is an n -equivalence, weak equivalence, or homotopy equivalence.*

We have used the following easy consequence.

Lemma 6.2. *Let $F : \mathcal{C} \rightarrow \mathcal{T}$ be a functor such that $F(\alpha)$ is a weak equivalence for every morphism α in \mathcal{C} and let \mathcal{C} have an initial object $\underline{0}$. Then the inclusion*

$$F(\underline{0}) \rightarrow \mathrm{hocolim}_{\mathcal{C}} F$$

is a weak equivalence.

Proof. Let E be the constant functor that takes the value $F(\underline{0})$. By inspection of definitions, $\mathrm{hocolim}_{\mathcal{C}} E$ is homeomorphic to $F(\underline{0}) \wedge BC_+$, and BC is contractible since J has an initial object. Therefore the inclusion $F(\underline{0}) \rightarrow \mathrm{hocolim}_{\mathcal{C}} E$ is a homotopy equivalence. Applying F to the unique map from $\underline{0}$ to each object of \mathcal{C} , we obtain a natural transformation $\phi : E \rightarrow F$. By the previous lemma, the induced map $\phi_* : \mathrm{hocolim}_{\mathcal{C}} E \rightarrow \mathrm{hocolim}_{\mathcal{C}} F$ is a weak equivalence. Clearly ϕ_* restricts to the inclusion on $F(\underline{0})$. \square

Homotopy colimits are also functorial in \mathcal{C} . Given indexing categories \mathcal{C} and K and a functor $f : \mathcal{C} \rightarrow K$, we obtain an induced map

$$f_* : \mathrm{hocolim}_{\mathcal{C}} F \circ f \rightarrow \mathrm{hocolim}_K F.$$

We have used the following result relating natural transformations to homotopies.

Lemma 6.3. *Let $\eta : f \rightarrow g$ be a natural transformation between functors $f, g : \mathcal{C} \rightarrow K$. Then the following diagram is naturally homotopy commutative:*

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathcal{C}} F \circ f & \xrightarrow{(F\eta)_*} & \mathrm{hocolim}_J F \circ g \\ & \searrow f_* \quad \swarrow g_* & \\ & \mathrm{hocolim}_K F & \end{array}$$

Proof. Let I be the category with two objects $[0]$ and $[1]$ and one non-identity morphism $[0] \rightarrow [1]$. Then $BI \cong [0, 1]$. It is standard that η determines and is determined by a functor $\bar{\eta} : \mathcal{C} \times I \rightarrow K$ that restricts to f on $\mathcal{C} \times \{[0]\}$ and to g on $\mathcal{C} \times \{[1]\}$. Let $\pi : \mathcal{C} \times I \rightarrow \mathcal{C}$ be the projection. Then η also determines a natural transformation $\tilde{\eta} : f \circ \pi \rightarrow \bar{\eta}$. For an object c of \mathcal{C} , $\tilde{\eta}(c, [0]) = \text{id} : f(c) \rightarrow f(c)$ and $\tilde{\eta}(c, [1]) = \eta : f(c) \rightarrow g(c)$. The required natural homotopy $h : f_* \simeq g_* \circ (F\eta)_*$ is the composite

$$\begin{aligned} (\text{hocolim}_{\mathcal{C}} F \circ f) \times BI &\cong \text{hocolim}_{\mathcal{C} \times I} F \circ f \circ \pi \\ &\xrightarrow{(F\tilde{\eta})_*} \text{hocolim}_{\mathcal{C} \times I} F \circ \bar{\eta} \\ &\xrightarrow{\bar{\eta}_*} \text{hocolim}_K F, \end{aligned}$$

where the first isomorphism is a direct inspection of definitions. \square

We now turn to the proof of Proposition 4.5 which depends on the expression of homotopy colimits over I as homotopy colimits over certain subcategories of I . Let I_n be the full subcategory of I consisting of the objects \mathbf{m} for $m \geq n$ and let f_n be the inclusion of I_n in I .

Lemma 6.4. *Let $F : I \rightarrow \mathcal{T}$ be a functor. For each n , the map*

$$(f_n)_* : \text{hocolim}_{I_n} (F \circ f_n) \rightarrow \text{hocolim}_I F$$

induced by f_n is a homotopy equivalence.

Proof. Define a functor $g_n : I \rightarrow I_n$ by concatenation with \mathbf{n} ; explicitly, $g_n(\mathbf{m}) = \mathbf{m} + \mathbf{n}$ and $g_n(i) = i \sqcup \text{id}_{\mathbf{n}}$ for an injection $i : \mathbf{m} \rightarrow \mathbf{m}'$. Let $z_n : \mathbf{0} \rightarrow \mathbf{n}$ be the unique map, define $\zeta(\mathbf{m}) = \text{id}_{\mathbf{m}} \sqcup z_n : \mathbf{m} \rightarrow \mathbf{m} + \mathbf{n}$, and define $\eta(\mathbf{m}) = \zeta(\mathbf{m})$ for $m \geq n$. Then ζ is a natural transformation $\text{Id}_I \rightarrow f_n \circ g_n$, η is a natural transformation $\text{Id}_{I_n} \rightarrow g_n \circ f_n$, and $f_n \eta = \zeta f_n : f_n \rightarrow f_n \circ g_n \circ f_n$. Via two applications of Lemma 6.3 and a little diagram chasing, we find that the composite

$$\text{hocolim}_I F \xrightarrow{(F\zeta)_*} \text{hocolim}_I F \circ f_n \circ g_n \xrightarrow{(g_n)_*} \text{hocolim}_{I_n} F \circ f_n$$

is a homotopy inverse to $(f_n)_*$. \square

Proof of Proposition 4.5. It suffices to show that ϕ_* is an N -equivalence for each $N > 0$. Choose n such that $\lambda(n) > N$. Then for every object \mathbf{m} in I_n the map $\phi : F(\mathbf{m}) \rightarrow F'(\mathbf{m})$ is an N -equivalence. By Lemma 6.1, this implies that

$$\phi_* : \text{hocolim}_{I_n} F \circ f_n \rightarrow \text{hocolim}_{I_n} F' \circ f_n$$

is an N -equivalence. The proposition now follows from Lemma 6.4. \square

For a based space X , we define $F \wedge X$ by setting $(F \wedge X)(c) = F(c) \wedge X$ and have a natural homeomorphism

$$\text{hocolim}_{\mathcal{C}} (F \wedge X) \cong (\text{hocolim}_{\mathcal{C}} F) \wedge X.$$

We define homotopy colimits of symmetric spectra levelwise. A functor $F : \mathcal{C} \rightarrow Sp^{\Sigma}$ restricts to give functors $F(\mathbf{n}) : \mathcal{C} \rightarrow \mathcal{T}$ for each $n \geq 0$, and

$$(\text{hocolim}_{\mathcal{C}} F)(\mathbf{n}) = \text{hocolim}_{\mathcal{C}} F(\mathbf{n}),$$

and since homotopy colimits commute with the smash product $\text{hocolim}_{\mathcal{C}} F$ is a symmetric spectrum. All of the statements above remain valid for homotopy colimits of symmetric spectra.

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